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Boundary-layer separation on a sphere in a rotating flow

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The velocity just outside the boundary layer and upstream of the separation ring on a sphere moving along the axis of a slightly viscous, rotating fluid is calculated through a least-squares approximation on the hypothesis of no upstream influence. A reverse flow is found in the neighbourhood of the forward stagnation point for $k \equiv 2\Omega a/U > k_* = 2{\cdot}20$ (Ω = angular velocity of fluid, U = translational velocity of sphere, a = radius of sphere) and is accompanied by a forward-separation bubble, such as that observed by Maxworthy (1970) for $k \gtrsim 1$. Rotation also induces a downstream shift of the peak velocity; the estimated shift of the separation ring in the absence of forward separation increases with k to a maximum of 24°, in qualitative agreement with Maxworthy's observations.

The least-squares formulation is compared with that given by Stewartson (1958) for unseparated flow (Stewartson did not consider separation). Both formulations require truncation of an infinite set of simultaneous equations, but Stewartson's formulation yields a non-positive-definite matrix that may exhibit spurious singularities. The least-squares formulation yields a positive-definite matrix, albeit at the expense of slower convergence for fixed k, and is especially well suited for automatic computation.

An *ad hoc* incorporation of a cylindrical wave of strength $\mathscr{U}$, such that the maximum upstream axial velocity is $\mathscr{U}U$, is considered in an appendix. It is found that k_* decreases monotonically from 2·2 to 0 as $\mathscr{U}$ increases from 0 to 1.

1. Introduction

Forward separation of the axisymmetric flow past a body in a rotating liquid and the subsequent formation of an upstream separation bubble, as recently observed by Maxworthy (1970), now appears as perhaps the most challenging aspect of a well-known, but still controversial, problem. We attack that problem by calculating the peripheral velocity, with special reference to laminar boundary-layer separation, on a sphere moving along the axis of a slightly viscous, unbounded, rotating liquid (see figure 1). The basic parameter for this flow is the inverse Rossby number,

$$k = 2\Omega a/U, \tag{1.1}$$

where Ω is the angular velocity of the (undisturbed) liquid, U is the translational velocity of the sphere, and a is its radius.

Maxworthy reports that the flow past a sphere in a slightly viscous, rotating flow separates downsteam of the equator for all k (within the range of his observations), much as in a non-rotating flow, but that the (aft) separation ring shifts downstream as k increases from 0 to 2; he does not report quantitative measurements of this shift, but his photographs suggest that it is between 20 and 30° for $k = 2$. He also reports that forward separation occurs for $k \gtrsim 1$ and leads to the formation of a separation bubble (forward separation may occur for $k < 1$, but, if so, the separation bubble is not prominent), for which there is no counterpart in a non-rotating flow.

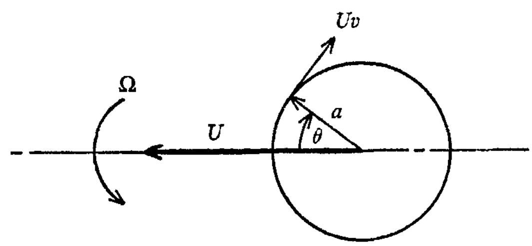

FIGURE 1. Sphere in rotating flow.

We find that the principal effects of rotation on the peripheral velocity in an inviscid, unseparated flow are to reduce the stagnation-point acceleration and to shift the peak velocity downstream. The former effect implies a region of reverse flow, and hence a forward-separation bubble, in the neighbourhood of the stagnation point for $k > k_*$, where $k_* = 2{\cdot}20$ on the hypothesis of uniform upstream flow. The downstream shift of the peak velocity, and hence of the domain of unfavourable pressure gradient (in the absence of stagnation-point reversal), implies a downstream shift of the separation ring that increases with k to an estimated maximum of 24° (this shift could be much larger for flows with forward separation).

The primary difficulty in the calculation of the inviscid, rotating flow over a *prescribed* surface is the prescription of the upstream conditions (we use *upstream* to imply a distance forward of the surface that is much larger than U/Ω). The hypothesis of uniform upstream flow holds for unseparated flow over a closed surface (Miles 1970*a*) and yields Long's (1953) model.† This model fails for separated flow; in particular, it does not hold within the domain of closed stream surfaces and therefore is inadequate for a complete description of flow with forward separation. [Long's model is characterized by simple proportionality between the total (Stokes) stream function and the azimuthal circulation, say Γ, and between the perturbation stream function and the product, say χ, of the azimuthal vorticity and the cylindrical radius, r. Inviscid flow within a domain of

† The hypothesis of uniform upstream flow implies that the particles in the unseparated flow on the surface of the sphere originate on the upstream axis, and hence that the sphere does not rotate in the laboratory reference frame (with respect to which the basic flow is rotating with the angular velocity Ω). Maxworthy's (1970) observations reveal that the sphere, although unconstrained by external torques, does not rotate (so that the velocity just outside the boundary layer is purely meridional) for $k \lesssim 1$ and rotates quite slowly, if at all, for $1 \lesssim k \lesssim 2$.

closed stream surfaces under the conditions considered herein is characterized by Γ = const. and χ = const. $\times\, r^2$ (Batchelor 1956).]

The problem of a sphere in an inviscid rotating flow goes back to Taylor (1922), but the first complete solution is due to Stewartson (1958). Stewartson provisionally adopts Long's model but is especially concerned with the circumstances under which a cylindrical flow (Taylor column) might exist. He does not, however, consider the possibility of a forward-separation bubble (which, to be sure, had not been observed at the time).

Stewartson finds that the wave drag increases rapidly with k and conjectures that the flow cannot be uniform upstream, and that a cylindrical flow must occur, for sufficiently large k. His original (1958) calculations indicate that the wave drag is infinite for $k = 5{\cdot}76$, but this appears to be a spurious singularity (Miles 1969). Stewartson's (1969) subsequent calculations yield finite (although still steeply rising with k) values of the wave drag for $k \leqslant 6$, and he asserts that the earlier difficulty is "essentially due to retaining an insufficient number of significant figures in the computations". The fact remains, however, that his basic formulation is characterized by a non-positive-definite matrix that may exhibit spurious singularities for sufficiently large k; it therefore appears desirable to obtain a more definitive formulation that establishes the non-existence of singular values of k.

We avoid the possibility of spurious singularities by a least-squares formulation that is characterized by a positive-definite matrix. This formulation yields a less rapidly convergent sequence of approximations for fixed k than does Stewartson's formulation (see appendix), but it is better suited for automatic computation, and we obtain definitive numerical results within the parametric range of interest. It also may be of interest for other problems in fluid mechanics.

The available theory of bluff-body flows is, of course, inadequate for a complete description of separation even in the absence of rotation. The essential features of the observed flow past a sphere in the absence of rotation ($k = 0$) are that: the peripheral velocity is qualitatively similar to that implied by potential theory upstream of the boundary-layer separation ring, but is quantitatively similar thereto only in the neighbourhood of the forward stagnation point; (laminar) separation occurs just upstream of the equator. For example, Fage's measurements (Rosenhead 1963, p. 423) yield

$$v = \tfrac{3}{2}(\theta - 0{\cdot}2914\theta^3 + 0{\cdot}0987\theta^5 - 0{\cdot}0282\theta^7) \quad (0 \leqslant \theta \leqslant 1{\cdot}48), \tag{1.2}$$

where θ is the meridional angle measured from the upstream axis; potential theory implies the sinusoidal distribution

$$v = \tfrac{3}{2}(\theta - 0{\cdot}1667\theta^3 + 0{\cdot}0083\theta^5 - 0{\cdot}0002\theta^7 + \ldots). \tag{1.3}$$

The calculated position of the separation ring is 84° on the basis of (1.2) (Tomotika & Imai; cited by Rosenhead 1963) and 110° on the basis of (1.3) (Schlichting 1955, p. 167); Fage's measured value is 83°. These discrepancies between theory and observation in non-rotating flows must be borne in mind in assessing the present attempt to explain rotation-induced effects on separation.

2. Formulation of inviscid problem

We consider (see figure 1) the uniform translation, with velocity U, of a sphere of radius a along the axis of an inviscid, unbounded, rotating flow, refer all lengths and velocities to a and U, respectively, define the spherical polar coordinates R and θ such that $R = 1$ on the sphere and $\theta = 0$ on the upstream axis, and derive the velocity from a vector potential according to

$$\mathbf{v} = \nabla \times \boldsymbol{\phi} + k\boldsymbol{\phi}, \quad \boldsymbol{\phi} = \boldsymbol{\phi}_1 \phi(R,\theta), \tag{2.1}$$

where $\boldsymbol{\phi}_1$ is a unit vector in the azimuthal direction, $r\phi$ is the Stokes stream function, and $r = R\sin\theta$ is the cylindrical radius. We neglect the effects of the downstream wake on the inviscid flow. The potential function ϕ then satisfies the wave equation [cf. Batchelor (1967, §7.5), in whose notation $\psi = r\phi$, $C = k\psi$, and $dH/d\psi = \frac{1}{2}k^2$]

$$(\nabla^2 + k^2 - r^{-2})\phi = \tfrac{1}{2}k^2 r, \tag{2.2}$$

the boundary condition (the stream function vanishes on the sphere)

$$\phi = 0 \quad (R = 1), \tag{2.3}$$

and the upstream condition (no inertial waves appear in the upstream flow)

$$\phi \sim \phi_\infty(r) + o(R^{-1}) \quad (R \to \infty, 0 \leqslant \theta < \tfrac{1}{2}\pi), \tag{2.4}$$

where

$$\phi_\infty = \tfrac{1}{2}r \tag{2.5}$$

is a particular solution of (2.2) that corresponds to the assumed upstream flow.† The velocity on the sphere is given by

$$v(\theta) = R^{-1}\partial(R\phi)/\partial R \tag{2.6a}$$

$$= \partial\phi/\partial R \quad (R = 1), \tag{2.6b}$$

where (2.6b) follows from (2.6a) by virtue of (2.3).

The general solution of (2.2) has the asymptotic form

$$\phi - \phi_\infty \sim (kR)^{-1}\mathscr{R}\{f(\theta)\,e^{ikR}\} + O(R^{-2}) \quad (kR \to \infty), \tag{2.7}$$

where $f(\theta)$ is the complex scattering amplitude of the inertial-wave field and must vanish identically in $0 \leqslant \theta < \frac{1}{2}\pi$ if (2.4) is to be satisfied exactly. A mean-square measure of the error in any approximate solution that satisfies (2.3) is provided by the *upstream scattering coefficient* (which is a measure of the energy in the upstream wave field),

$$\epsilon = \lim_{R\to\infty} R^2 \int_0^{\frac{1}{2}\pi} |\mathbf{v} - \mathbf{v}_\infty|^2 \sin\theta\, d\theta = \int_0^{\frac{1}{2}\pi} |f(\theta)|^2 \sin\theta\, d\theta. \tag{2.8}$$

3. Least-squares approximation

A particular solution of (2.2) that yields uniform upstream flow and $\mathbf{v} = 0$, i.e. $\phi = v = 0$ rather than only $\phi = 0$, on the sphere is given by [this curious solution is due to Taylor (1922)]

$$\phi_0(R,\theta) = \tfrac{1}{2}r - \tfrac{1}{2}k^2\{j_2 y_1(kR) - y_2 j_1(kR)\}\sin\theta, \tag{3.1}$$

† We consider a more general upstream condition in appendix B.

where j_n and y_n are spherical Bessel functions, the argument of each of which is implicitly k except where it is explicitly displayed as kR. We follow the definitions of Antosiewicz (Abramowitz & Stegun 1964) and refer to formulas therein by the prefix AS. That ϕ_0 satisfies (2.3) follows from AS 10.1.31; that it satisfies $\partial\phi_0/\partial R = 0$ at $R = 1$ follows from AS 10.1.22 and 10.1.31.

The most general solution of (2.2) that yields uniform upstream flow and satisfies (2.3) is given by

$$\phi(R,\theta) = \phi_0(R,\theta) + k \sum_{n=1}^{\infty} V_n\{j_n y_n(kR) - y_n j_n(kR)\} P_{n1}(\cos\theta), \tag{3.2}$$

where
$$P_{n1}(\mu) \equiv (1-\mu^2)^{\frac{1}{2}} dP_n(\mu)/d\mu = -P_n^1(\mu) \quad (\mu = \cos\theta) \tag{3.3}$$

is an associated Legendre function. Substituting (3.2) into (2.6b) and invoking the Wronskian of j_n and y_n (AS 10.1.6), we obtain

$$v(\theta) = \sum_{n=1}^{\infty} V_n P_{n1}(\cos\theta). \tag{3.4}$$

We determine the V_n approximately by truncating the expansions of (3.2) and (3.4) at, say, $n = N$, and minimizing ϵ with respect to each of $V_1, \ldots, V_N$. Subtracting $\phi_\infty = \frac{1}{2}r$ from (3.2) and truncating the expansion, we place the result in the form

$$\phi - \phi_\infty = \mathscr{R} \sum_{n=1}^{\infty} \{\delta_{1n} f_0(\mu) - V_n f_n(\mu)\} i^{n+1} h_n^{(1)}(kR), \tag{3.5}$$

where
$$f_0(\mu) = -\tfrac{1}{2} i k^2 h_2^{(2)} P_{11}(\mu), \quad f_n(\mu) = i^{-n} k h_n^{(2)} P_{n1}(\mu), \tag{3.6a,b}$$

δ_{1n} is the Kronecker delta, and $h_n^{(1,2)} = j_n \pm i y_n$ are spherical Hankel functions. Invoking the asymptotic approximation

$$h_n^{(1)}(kR) \sim (-i)^{n+1}(kR)^{-1} e^{ikR} \quad (kR \to \infty)$$

in (3.5) and comparing the result to (2.7), we obtain

$$f(\theta) = f_0(\mu) - \sum_{n=1}^{N} V_n f_n(\mu). \tag{3.7}$$

Substituting (3.7) into (2.8), we obtain the positive-definite, inhomogeneous quadratic form

$$\epsilon = \epsilon_0 - 2\mathscr{R} \sum_{n=1}^{N} C_n V_n + \sum_{m=1}^{N} \sum_{n=1}^{N} S_{mn} V_m V_n, \tag{3.8}$$

where
$$\epsilon_0 = \int_0^1 |f_0|^2 d\mu = \tfrac{1}{8} k^2 I_1 D_{22}, \tag{3.9}$$

$$C_n = \int_0^1 f_0^* f_n \, d\mu = \tfrac{1}{4} k(\delta_{1n} I_1 + i I_{1n})(D_{2n} - i E_{2n}), \tag{3.10}$$

$$S_{mn} = \int_0^1 f_m^* f_n \, d\mu = \tfrac{1}{2}(\delta_{mn} I_n + i I_{mn})(D_{mn} - i E_{mn}), \tag{3.11}$$

the asterisk implies complex conjugation,

$$I_n = 2\int_0^1 P_{n1}^2(\mu)\, d\mu = 2n(n+1)(2n+1)^{-1}, \tag{3.12}$$

$$I_{mn} = -I_{nm} = 2i^{m-n-1}\int_0^1 P_{m1}(\mu)P_{n1}(\mu)\,d\mu \quad (m \neq n) \tag{3.13a}$$

$$= 0\,(m-n \text{ even}) \tag{3.13b}$$

$$= \frac{2^{2-m-n}m(m+1)!\,n!}{(m-n)(m+n+1)[(\frac{1}{2}m)!\,(\frac{1}{2}n-\frac{1}{2})!]^2} \quad (m \text{ even}, n \text{ odd}), \tag{3.13c}$$

$$D_{mn} = k^2(j_m j_n + y_m y_n) = D_{nm}, \tag{3.14}$$

and

$$E_{mn} = k^2(j_m y_n - j_n y_m) = -E_{nm}. \tag{3.15}$$

Minimizing the right-hand side of (3.8) with respect to V_n for $n = 1, 2, \ldots, N$, we obtain the matrix equation

$$[\delta_{mn} I_n D_{nn} + I_{mn}E_{mn}]\{V_n\} = \tfrac{1}{2}k\{\delta_{1m}I_1 D_{12} + I_{1m}E_{2m}\}. \tag{3.16}$$

We remark that both D_{mn} and E_{mn} are algebraic functions of k with recurrence relations that may be inferred from those for j_n and y_n. In particular, $D_{12} = (2/k) + (3/k^3)$, D_{nn} is given by AS 10.1.27, and E_{mn} may be calculated from

$$E_{mn} = 0 \quad (m = n) \tag{3.17a}$$

$$= 1 \quad (m = n+1) \tag{3.17b}$$

$$= (2m-1)k^{-1}E_{m-1,n} - E_{m-2,n} \quad (m \geqslant n+2). \tag{3.17c}$$

k	N	V_1	$-V_2$	V_3	v_0'	C_D	ϵ
1	2	1·285	0·0618	—	1·099	0·2539	0·0254
	4	1·291	0·0621	$1{\cdot}00.10^{-4}$	1·100	0·2566	0·0155
	8	1·296	0·0623	$1{\cdot}01.10^{-4}$	1·103	0·2583	0·0088
	20	1·299	0·0625	$1{\cdot}02.10^{-4}$	1·106	0·2596	0·0039
	∞	1·301	0·0626	$1{\cdot}02.10^{-4}$	1·108	0·2606	$1{\cdot}5.10^{-5}$
2	2	1·454	0·3929	—	0·2750	1·827	0·2092
	4	1·566	0·4295	0·0232	0·2114	2·211	0·1360
	8	1·651	0·4537	0·0254	0·2159	2·464	0·0823
	20	1·721	0·4731	0·0270	0·2242	2·680	0·0381
	40	1·749	0·4810	0·0277	0·2276	2·770	0·0201
	60	1·759	0·4838	0·0280	0·2288	2·803	0·0137
	∞	1·777	0·4888	0·0284	0·2309	2·862	0·0043
3	8	2·75	1·33	0·292	−0·76	14·1	0·352
	20	3·22	1·57	0·368	−0·97	19·9	0·198
	40	3·47	1·70	0·408	−1·08	23·4	0·115
	∞	3·73	1·83	0·448	−1·19	27·1	0·024
4	8	4·4	2·9	1·0	−0·4	61	0·84
	20	5·8	3·9	1·6	−1·2	117	0·62
	40	7·1	4·8	2·0	−1·9	179	0·45
	∞	8·2	5·7	2·5	−2·5	251	0·11

TABLE 1. The convergence of the solution to (3.16) with increasing N. The results labelled ∞ are determined by a least-squares extrapolation of V_n *versus* $1/N$ for $N = 20(4)40$.

The numerical convergence of the solution of (3.16) is illustrated in table 1, where V_1, V_2, V_3, ϵ, the stagnation-point acceleration,

$$v_0' = \frac{1}{2}\sum_{n=1}^{N} n(n+1)V_n, \tag{3.18}$$

and the wave-drag coefficient (on the hypothesis of unseparated flow),

$$C_D = \frac{\text{drag}}{\frac{1}{2}\pi\rho U^2 a^2} = -2\int_{-1}^{1} v^2 \mu \, d\mu = -8 \sum_{n=1}^{N-1} \frac{n(n+1)(n+2)}{(2n+1)(2n+3)} V_n V_{n+1}, \tag{3.19}$$

are tabulated. The results labelled ∞ were determined by extrapolating a least-squares fit to V_n *versus* $1/N$, as determined by (3.16) for $N = 20(4)40$ (the error in a straight-line interpolation of these points is typically less than 0·01 %!). It appears from these results (although we have not proved) that the Nth approximation to $|V_n|$ tends monotonically toward the exact result from below as $N \to \infty$ and that the convergence is highly non-uniform as $k \to \infty$ (much as in the solution of plane-wave diffraction by a sphere through a modal expansion). The results for $V_{1,2,3}$ are plotted in figure 2, those for v_0' and C_D in figure 3. The peripheral velocity distributions are plotted in figure 4 (these distributions were determined for $N = 4$ and are less accurate than the preceding results).

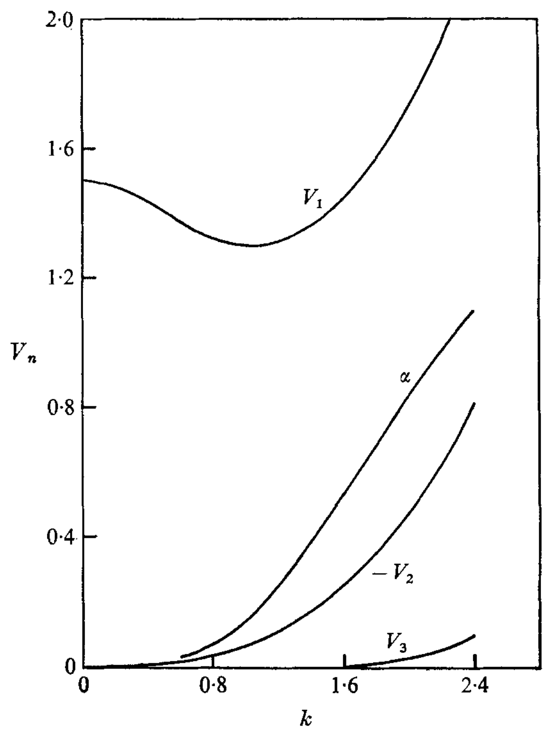

FIGURE 2. The expansion coefficients $V_{1,2,3}$ of (3.4) as determined by (3.16). The parameter $\alpha = -3V_2/V_1$ appears in the approximation (4.3).

4. Boundary-layer separation

It is evident from the results of the preceding section that rotation generally decreases the stagnation-point acceleration and implies the reversal of the flow in a finite neighbourhood of the stagnation point for $k > k_*$, where $k_* = 2{\cdot}20$ on the hypotheses of uniform upstream flow and unseparated flow over the entire sphere. This reversed flow is necessarily accompanied by a forward separation bubble such as that observed by Maxworthy (1970), albeit at values of k significantly smaller than 2·2. There also are one or more reversals in the outer flow for $k > k_c$, where $k_c \doteqdot 2{\cdot}2$ (Miles 1969; the agreement between the estimates of k_* and k_c appears to be coincidental). These outer reversals appear to imply at least local instability of the flow in consequence of the concomitant violation of Rayleigh's criterion that the square of the circulation must be a monotonically increasing function of the cylindrical radius for stable flow. The theoretical model of §§2 and 3 cannot give a valid description of the flow within the forward separation bubble† or other regions of closed stream surfaces and appears to be

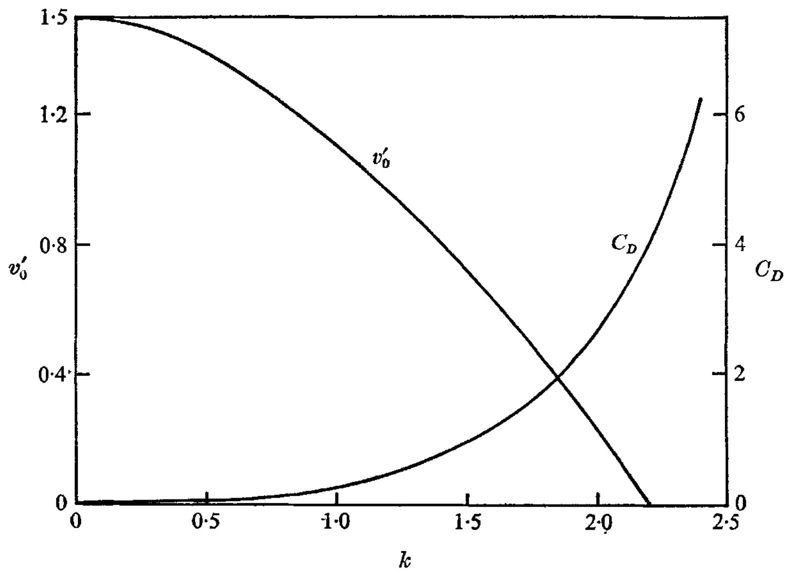

FIGURE 3. The stagnation-point acceleration and wave-drag coefficient, as determined by (3.18) and (3.19).

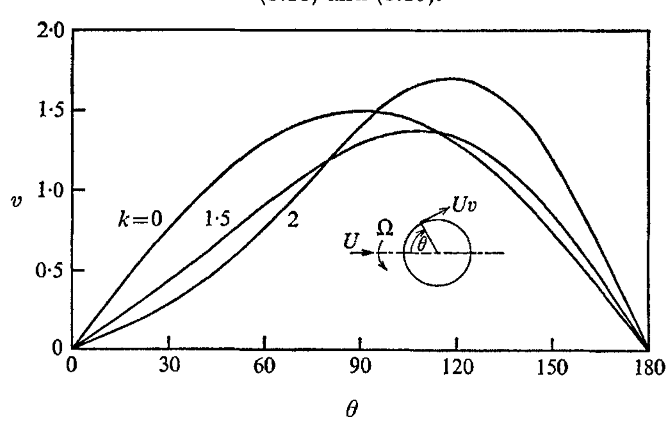

FIGURE 4. The peripheral velocity distributions for $N = 4$ (the maximum truncation error for $k = 2$ is roughly 10 %, in contrast to the negligible truncation errors for the results presented in figures 2 and 3).

† The actual flow inside the separation bubble for $k > k_*$ may be significantly affected by boundary-layer effects and may not be reversed. Professor Maxworthy informs me that his observations give no indication of a reversed flow.

adequate for the calculation of the flow upstream of the forward separation bubble if and only if this bubble is taken as the prescribed stream surface in the calculation of the outer flow (Miles 1969; Maxworthy 1970).

The second, general prediction of the preceding results is that rotation induces a downstream shift of the maximum peripheral velocity. Invoking the fact that the swirl implied by the inviscid solution vanishes on the sphere, we may estimate the corresponding shift of the separation ring, in that parametric régime in which forward separation does not occur, just as in a non-rotating flow with the same peripheral velocity, $v(s)$. We assume laminar flow and describe the boundary layer by the shape parameter

$$\lambda(s) \equiv (Ua/\nu)v'(s)\delta_2^2(s), \tag{4.1}$$

where $a\delta_2(s)$ is the momentum thickness, and s is the peripheral distance from the stagnation point. Invoking the simplified form of the momentum equation, as developed by Thwaites and Curle & Skan for two-dimensional flow and modified by Rott & Crabtree for axisymmetric flow (Rosenhead 1963, VI. 18 and VIII. 12), we obtain

$$\lambda(s) = 0{\cdot}45v'(s)v^{-6}(s)r^{-2}(s)\int_0^s v^5r^2\,ds. \tag{4.2}$$

We proceed on the basis of the approximation

$$v(\theta) = V_1(1-\alpha\cos\theta)\sin\theta \quad (\alpha \equiv -3V_2/V_1, 0 < \alpha < 1), \tag{4.3}$$

which corresponds to $N = 2$ in §3 and should be qualitatively adequate for that parametric régime in which forward separation does not occur [it seems likely that forward separation increases the downstream shift of the aft separation ring, but neither (4.3) nor the description of the boundary layer by the shape parameter λ is likely to be adequate for flows with forward separation]. Substituting $s = \theta$, $r = \sin\theta$, the approximation (4.3), and the change of variable $\mu = \cos\theta$ into (4.2), we obtain

$$\lambda = 0{\cdot}45(1-\mu^2)^{-4}(1-\alpha\mu)^{-6}[\mu+\alpha(1-2\mu^2)]\int_\mu^1 (1-\mu^2)^3(1-\alpha\mu)^5\,d\mu. \tag{4.4}$$

This last result is plotted in figure 5. The value of θ at separation, based on Curle & Skan's criterion, $\lambda = -0{\cdot}09$,† is plotted in the insert of figure 4; it increases from 103° to 127° as α increases from 0 to 1. The absolute values of these positions are not likely to be accurate (see last paragraph in §1), but it does not appear unreasonable to expect that the dependence on α of the relative shift implied by (4.4) is at least qualitatively correct.

We emphasize that the results of the preceding paragraph depend on the inviscid flow only through the parameter α and are independent of the hypotheses underlying the calculation of this parameter in §§2 and 3. It is conceivable

† It is curious, and perhaps more than coincidental, that the parameters a and b, defined by the empirical representation $F(\lambda) = a-b\lambda$ used by Thwaites (Rosenhead, p. 305), appear to be related according to $a = (b-1)/(2b-1)$, such that $F = 1$, $H = b-\frac{5}{2}$, and $T = 0$ at the separation point defined by $\lambda_s = -1/(2b-1)$. Choosing $b = 6$ [as does Thwaites and as is strongly suggested by the energy integral (Rosenhead, VI. 19)] yields $a = 5/11$ and $\lambda_s = -1/11$.

that other hypotheses could lead to an outer velocity of the form (4.3), but in which V_1 and α could exhibit quite different dependencies on k than those given in §3. It remains true, nevertheless, that the results of §3 do yield qualitative agreement with Maxworthy's observations of the effects of rotation on boundary-layer separation in that they imply the reversal of the stagnation-point acceleration for sufficiently large k and a downstream shift of the aft separation ring with increasing k.

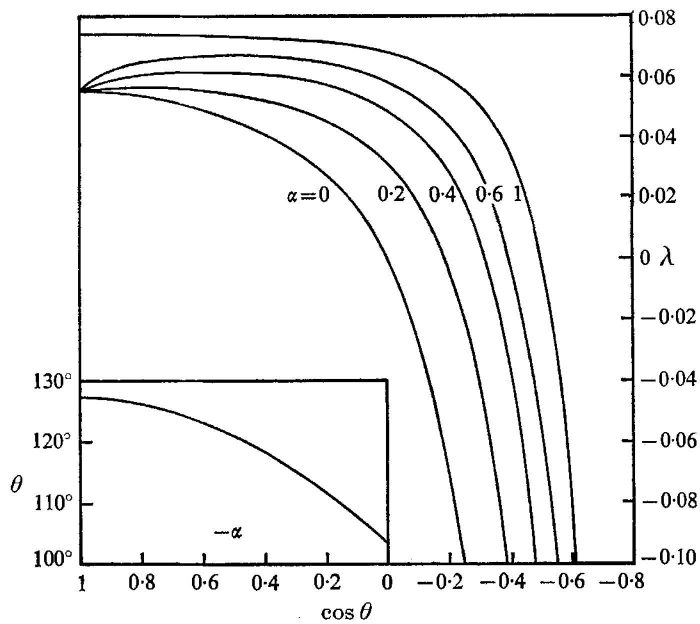

FIGURE 5. The boundary-layer shape parameter, as defined by (4.1) and approximated by (4.4), and the location of the separation ring, as determined by $\lambda = -0{\cdot}09$.

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Appendix A. Galerkin approximation

We consider the approximate determination of $V_1, \ldots, V_N$ in the truncated expansion of (3.5) by Galerkin's method. Substituting the approximation (3.7) into (2.7) and invoking (2.4), we obtain

$$\sum_1^N V_n f_n(\mu) \cong f_0(\mu) \quad (0 < \mu \leqslant 1). \tag{A 1}$$

Multiplying (A 1) through by $w_m(\mu)$, $m = 1, \ldots, N$, where $\{w_m(\mu)\}$ is a linearly independent set of weighting functions, and integrating over $0 < \mu < 1$, we obtain the matrix equation

$$[S_{mn}]\{V_n\} = \{C_m\}, \tag{A 2}$$

where $$S_{mn} = \int_0^1 w_m(\mu) f_n(\mu)\,d\mu \quad \text{and} \quad C_m = \int_0^1 w_m(\mu) f_0(\mu)\,d\mu. \tag{A 3a,b}$$

The real and imaginary parts of (A 2) comprise $2N$ real equations in the real and imaginary parts of $V_1, \ldots, V_N$. In fact, the V_n must be real if the approximation of (3.5) is to satisfy the boundary condition (2.3); accordingly, N of these $2N$ equations are redundant [this redundancy appears to be associated with the fact that each of the odd and even subsets of $P_{n1}(\mu)$ is complete in $0 < \mu < 1$].

The method of least squares corresponds to the choice $w_n = f_n^*$ and the retention of only the real part of (A 2), which then reduces to (3.16).

Stewartson (1958) attacks the boundary-value problem posed by (2.2)–(2.4) by first constructing a complete set of solutions, ψ_n/r below, that individually satisfy (2.4); he then applies Galerkin's method to solve a truncated approximation to (2.3). Referring to equations in Stewartson's paper by the prefix S, we find that the perturbation stream function given by S(4.2) and S(4.3) is equivalent to

$$r\phi - \tfrac{1}{2}r^2 = \tfrac{1}{2}\sum_{n=1}^{\infty} \hat{A}_n \psi_n(aR, \theta), \tag{A 4}$$

where $$\psi_n = (-)^{n-1}(2k/\pi)^{\frac{1}{2}} r[y_n(kR)P_{n1}(\mu) + \sum_{s=1}^{\infty} (I_{ns}/I_s) j_s(kR) P_{s1}(\mu)] \tag{A 5}$$

in the present notation, and $\hat{A}_n \equiv -A_n$ in S(4.2). Comparing (A 4) to (3.2) in the neighbourhood of $R = 0$, we obtain

$$\hat{A}_n = (2\pi k)^{\frac{1}{2}}\{(-)^{n-1} V_n j_n - \tfrac{1}{2}k j_2 \delta_{1n}\}. \tag{A 6}$$

Substituting $A_n = -\hat{A}_n$ into S(4.6), we obtain

$$[\delta_{mn} I_n j_n y_n - I_{mn} j_m j_n]\{V_n\} = \tfrac{1}{2}k\{\delta_{1m} I_1 j_1 y_2 + I_{1m} j_2 j_m\}, \tag{A 7}$$

which is identical with the matrix equation obtained by choosing

$$w_n = 2i^{n+1}k^{-1}j_n P_{n1}(\mu) \tag{A 8}$$

in (A 3) and taking the real part of (A 2).

We designate the formulations represented by (3.16) and (A 7) by I and II, respectively, and the corresponding square matrices by $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$. The matrix $\mathbf{S}_{\mathrm{I}}$ is positive definite and an algebraic function of k, by virtue of which I is well suited to high-speed computation (the matrix coefficients may be calculated from simple recursion formulas). The matrix $\mathbf{S}_{\mathrm{II}}$ is non-positive-definite and a transcendental function of k, in consequence of which it is less well suited to high-speed computation; in particular, the determinant $S_{\mathrm{II}}(k)$ may have zeros for $k > k_N$, where k_N is near the smallest zero of $y_N(k)$. [E.g. the smallest zeros of S_{II} for $N = 1$ and $N = 3$ are 2.8 and 4.3, respectively.] On the other hand, II appears to yield more rapid convergence (with increasing N) than I for $1 \lesssim k \ll k_N$. We give a numerical comparison of I and II (Stewartson 1968) for $N = 8$ in table 2, using as a basis of comparison the aforementioned extrapolation of the results given by I for $N = 20(4)40$. It is evident that II yields much smaller truncation errors in the quantities of principal interest for $k \leqslant 4$, even though it yields substantially larger values of upstream wave energy (as measured by ϵ); however, II yields much larger truncation errors for $k = 6$, presumably in consequence of the proximity of a zero of S_{II}. We also remark that II appears to yield upper bounds to the V_n, in contrast to the lower bounds determined by I.

k	N	V_1	$-V_2$	C_D	ϵ
1	8 (I)	1·296	0·0623	0·2583	0·0088
	8 (II)	1·302	0·06	0·260	0·0259
	∞	1·301	0·0626	0·2606	$1{\cdot}5.10^{-5}$
2	8 (I)	1·651	0·454	2·464	0·082
	8 (II)	1·781	0·489	2·872	0·166
	∞	1·777	0·489	2·862	−0·004
3	8 (I)	2·75	1·33	14·1	0·35
	8 (II)	3·83	1·89	28·8	1·10
	∞	3·73	1·83	27·1	0·02
4	8 (I)	4·4	2·9	61	0·84
	8 (II)	10·5	7·3	422	6·06
	∞	8·2	5·7	251	0·11
5	8 (I)	7	6	$2{\cdot}4.10^2$	1·4
	8 (II)	35	31	$8{\cdot}4.10^3$	—
	∞	15	12	$1{\cdot}3.10^3$	0·7
6	8 (I)	13	11	$1{\cdot}0.10^3$	2·0
	8 (II)	134	141	$2{\cdot}1.10^5$	—
	∞	29	27	$7{\cdot}2.10^3$	1·2

TABLE 2. Comparison of the results determined by (3.16) and Stewartson's (1958, 1969) formulation (note that Stewartson's C_D must be doubled to be compatible with the normalization adopted herein). The results labelled ∞ are determined by a least-squares extrapolation of V_n *versus* $1/N$ for $N = 20(4)40$.

Appendix B. Incorporation of cylindrical wave

The model developed in §§ 1 and 2 fails for separated flow, but the model of an oseenlet (Miles 1970*b*) suggests that the effect of the downstream wake on the inviscid flow outside of that wake might be represented by a cylindrical wave of strength $\mathscr{U}$, such that ϕ_∞ in § 2 is replaced by

$$\phi_\infty = \tfrac{1}{2}r - (\mathscr{U}/k)J_1(kr), \tag{B 1}$$

which implies†

$$u/U \sim 1 - \mathscr{U}J_0(kr) \quad (x \to -\infty). \tag{B 2}$$

We generalize the formulation of § 3 to incorporate the cylindrical wave of (B 1) by invoking the addition theorem [Watson (1945, § 11.5(9)) with $\nu = \frac{3}{2}$ and $\phi' = \frac{1}{2}\pi$; the terms for even n vanish identically]

$$J_1(kr) = \sum_{n=1}^{\infty} c_n j_n(kR)P_{n1}(\mu), \quad c_n = i^{n-1}n^{-1}(n+1)^{-1}(2n+1)P_{n1}(0), \tag{B 3a, b}$$

which provides a representation in terms of inertial waves of finite length (cf. the plane-wave expansion of classical diffraction theory).

† We emphasize that viscous effects cannot be neglected at sufficiently large distances from the sphere and that (B 2) is actually the leading term in the outer expansion of an inner approximation (see Miles 1970*b*).

Substituting (B 3) into (2.5), subtracting the result from (3.2), and truncating, we obtain (3.5) and (3.7) with V_n replaced by

$$V_n + i\mathscr{U}(k^2 h_n^{(2)})^{-1} c_n \equiv A_n + iB_n, \tag{B 4}$$

where $$A_n = V_n - \mathscr{U}(c_n y_n / D_{nn}) \text{ and } B_n = \mathscr{U}(c_n j_n / D_{nn}). \tag{B 5a, b}$$

We also obtain (3.8) with V_n and V_m replaced by $A_n + iB_n$ and $A_m - iB_m$, respectively. Minimizing the result with respect to $A_1, \ldots, A_N$, we obtain

$$[\delta_{mn} I_n D_{nn} + I_{mn} E_{mn}]\{A_n\} = \tfrac{1}{2}k\{\delta_{1n} I_1 D_{12} + I_{1n} E_{2n}\} + [I_{mn} D_{mn}]\{B_n\} \tag{B 6}$$

in place of (3.16). Substituting (B 5a) into (3.4) and truncating, we obtain

$$v(\theta) = \sum_{n=1}^{N} (A_n + \mathscr{U} c_n + D_{nn}^{-1} y_n) P_{n1}(\mu), \tag{B 7}$$

in which the series in the c_n is absolutely convergent as $N \to \infty$.

The set (B 7) was solved for $N = 4$; the estimated (on the basis of the more extensive investigation for $\mathscr{U} = 0$) truncation errors are less than 12(1) % for $k < 2(1)$. The principal effect of the cylindrical wave on boundary-layer separation, *vis-à-vis* the results for $\mathscr{U} = 0$, is the reduction of k_*, which decreases monotonically from 2.2 to 0 as $\mathscr{U}$ increases from 0 to 1. This may provide a partial explanation of Maxworthy's observations of separated flow for k as small as 1, but we emphasize that there is little or no observational support for (B 1) at this time [indeed, Maxworthy's observations suggest that the hypothesis of uniform upstream flow provides a good approximation to the inviscid flow upstream of the forward stagnation point (at the apex of the forward separation bubble) and may provide a useful approximation to the flow outside of that stream surface that comprises those fluid particles that originate on the upstream axis]. We also emphasize that our generalization incorporates $\mathscr{U}$ as an independent parameter, whereas a complete model presumably would yield $\mathscr{U}$ as a function of k.

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